



Computational Analysis of Heat Transfer in Two-Dimensional Physical Systems

¹Haemi Imamverde

¹Department of Electrical and Computer Engineering, Chiang Thon College of Management, Thailand

¹haemi.imamverde@ctcm-th.org

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Abstract

This methodology paper presents a computational framework for analyzing two-dimensional heat transfer in physical systems using a finite-difference formulation. A square isotropic solid with uniform internal heat generation is selected as a reproducible benchmark because it combines prescribed-temperature and convective boundary conditions and produces a genuinely two-dimensional temperature field. The steady heat-conduction equation is discretized on a structured Cartesian grid using second-order central differences. Dirichlet conditions are imposed at the left and right boundaries, while Robin convection conditions are introduced at the upper and lower boundaries by a ghost-node elimination procedure. The resulting sparse algebraic system is solved directly, and the computational solution is assessed through grid refinement and an overall energy-balance check. For the adopted benchmark, the center temperature converges to approximately 334.056 K, and refinement from 41×41 to 81×81 nodes change the center temperature by less than 0.001%. The net boundary heat-flow imbalance on the 81×81 grid is approximately 0.5% of the prescribed volumetric heat generation, supporting numerical consistency. The study demonstrates a transparent workflow for geometry definition, governing-equation formulation, discretization, solution, verification, and post-processing. The approach is suitable as a baseline for more complicated two-dimensional conduction, conjugate heat-transfer, and enclosure-flow problems. All literature cited in the methodological development was published in or before 2015.

Introduction

Heat transfer in two-dimensional systems occurs in a wide range of engineering applications, including electronic packages, thermal barriers, heat exchangers, building components, solar devices, and energy-storage structures. When geometry, material properties, heat generation, or boundary conditions vary in more than one spatial direction, one-dimensional analytical approximations can become inadequate. Computational heat-transfer methods therefore provide an important means of resolving temperature

gradients and quantifying heat flow while retaining the physical boundary conditions of the problem.

Benchmark calculations have played a central role in establishing trustworthy numerical heat-transfer practice. De Vahl Davis (1983) demonstrated the importance of second-order discretization, mesh refinement, and comparison of integral quantities in a classic two-dimensional square-cavity problem. Although that benchmark concerns buoyancy-driven flow rather than pure solid conduction,

its verification philosophy is directly transferable to computational thermal analysis. Later studies used finite-difference formulations for two-dimensional enclosure heat transfer with internal heated elements and showed that temperature and heat-transfer predictions are strongly dependent on the imposed thermal boundaries and grid resolution (Kandaswamy, Lee, & Abdul Hakeem, 2007; Selamat, Hashim, & Hasan, 2012).

At the same time, alternative discretization approaches expanded the range of tractable heat-conduction problems. Li, Chen, and Kou (2011) applied a meshless local Petrov-Galerkin strategy to transient heat conduction, while Marchi, Novak, Santiago, and Vargas (2013) demonstrated how Richardson extrapolation can quantify and reduce discretization error for the two-dimensional Laplace equation. Dai, Zheng, Liang, and Wang (2013) also developed an improved meshless formulation for two-dimensional transient conduction. These studies reinforce two methodological requirements: the governing thermal model must be discretized consistently, and numerical error must be examined rather than assumed to be negligible.

The objective of the present work is to formulate a concise and reproducible methodology for computational analysis of heat transfer in a two-dimensional physical system. A square plate with internal heat generation and mixed thermal boundary conditions is used as the model problem. The finite-difference method is selected because of its clarity, low implementation overhead, and suitability for structured domains. The method is then checked through mesh refinement and energy conservation before the resulting temperature field and heat-flow distribution are interpreted.

Literature Review

Numerical heat-transfer literature up to 2015 shows a progression from classical mesh-based discretization toward increasingly flexible meshless and stabilized methods. The square-cavity solution of De Vahl Davis (1983) remains influential because it established high-quality reference values through systematic refinement. Kandaswamy et al. (2007) considered natural convection around a heated plate inside a square enclosure using finite differences,

alternating-direction implicit treatment, and successive over-relaxation. Their work illustrated how localized thermal sources can reshape the two-dimensional temperature and flow fields.

For transient conduction, Li et al. (2011) combined the meshless local Petrov-Galerkin method with a precise time-integration strategy and emphasized accurate enforcement of essential boundary conditions. Selamat et al. (2012) solved transient natural convection in a porous square cavity by finite differences and an alternating-direction implicit procedure, again reporting grid- and convergence-related numerical controls. Marchi et al. (2013) focused directly on discretization error in the two-dimensional Laplace equation and showed that repeated Richardson extrapolation can substantially improve numerical accuracy when solutions are available on systematically refined grids.

Dai et al. (2013) proposed an improved meshless local Petrov-Galerkin scheme for two-dimensional unsteady heat conduction, demonstrating close agreement with analytical and finite-element results. Reséndiz-Flores and García (2014) applied a finite pointset method to non-stationary two-dimensional conduction and compared numerical predictions with analytical solutions. Hidayat, Wahjoedi, Parman, and Megat Yusoff (2014) developed a local B-spline finite-difference method for two-dimensional conduction with heat generation and spatially varying thermal conductivity, showing that generalized finite-difference ideas can accommodate irregular nodal distributions and material variation.

By 2015, the literature also emphasized computational efficiency. Huang, Zhao, and Feng (2015) presented a stabilized finite-element formulation for natural convection that reduced computational cost while retaining accuracy. Mazzearella and Pasini (2015) compared finite-difference and conduction-transfer-function methods for transient building heat conduction, highlighting the trade-off among stability, accuracy, and computational time. Taken together, these works support the use of a simple finite-difference baseline when the domain is structured, provided that discretization error, convergence, and heat balance are explicitly checked.

Table 1. Selected computational heat-transfer studies published up to 2015.

Study	Year	Method / system	Methodological relevance
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De Vahl Davis	1983	2D finite difference; square cavity	Benchmarking and mesh refinement
Kandaswamy et al.	2007	FD/ADI/SOR; heated plate cavity	Mixed thermal boundaries in 2D
Li et al.	2011	MLPG; transient conduction	Boundary treatment and time integration
Marchi et al.	2013	FD + Richardson extrapolation	Discretization-error estimation
Dai et al.	2013	Improved MLPG; 2D transient conduction	Meshless verification against analytical/FEM
Hidayat et al.	2014	Local B-spline FD; 2D conduction	Heat generation and variable conductivity
Mazzarella & Pasini	2015	FD vs CTF; transient conduction	Accuracy-stability-cost comparison

Methodology

Step 1 - Physical Domain and Assumptions

A square solid domain of side length $L = H = 0.10$ m is considered. The material is assumed homogeneous, isotropic, and characterized by a constant thermal conductivity $k = 16$ W m⁻¹ K⁻¹. Uniform volumetric heat generation $q''' = 20,000$ W m⁻³ occurs throughout the solid. The analysis is steady, two-dimensional, and free of radiative exchange. Thermal contact resistance is neglected. These assumptions isolate the spatial conduction mechanism while retaining mixed boundary conditions that force temperature variation in both x and y directions.

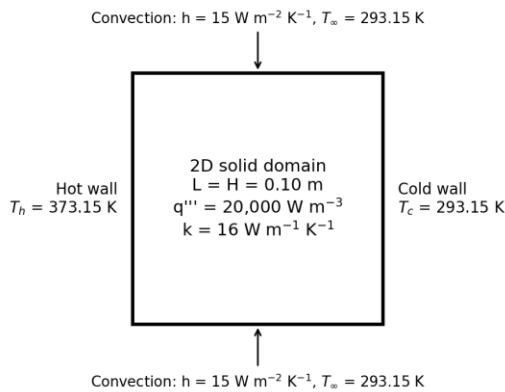


Figure 1. Computational domain and thermal boundary conditions used in the benchmark analysis.

Step 2 - Governing Equation

For steady conduction with constant thermal conductivity and uniform volumetric heat

generation, conservation of energy in Cartesian coordinates gives

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{q'''}{k} = 0. \quad (1)$$

Equation (1) is a Poisson equation. It reduces to the Laplace equation only when internal heat generation is zero. The present form therefore provides a more general thermal benchmark than a purely source-free plate and is consistent with the class of two-dimensional conduction models used in numerical verification studies (Marchi et al., 2013; Hidayat et al., 2014).

Step 3 - Boundary Conditions

The left wall is maintained at $T_h = 373.15$ K and the right wall at $T_c = 293.15$ K. The upper and lower boundaries exchange heat with an ambient at $T_\infty = 293.15$ K through a convection coefficient $h = 15$ W m⁻² K⁻¹. Accordingly, the boundary conditions are $T(0,y)=T_h$, $T(L,y)=T_c$, and $-k(\partial T/\partial n)=h(T-T_\infty)$ on the horizontal boundaries. The mixed Dirichlet-Robin specification creates vertical as well as horizontal gradients and avoids a trivial one-dimensional linear solution.

Step 4 - Spatial Discretization

The domain is divided into a uniform Cartesian mesh. At an interior node (i,j) , second-order central differences are applied to the two second derivatives. For equal spacings Δx and Δy , the discrete equation is

$$(T_{i+1,j} - 2T_{i,j} + T_{i-1,j})/\Delta x^2 + (T_{i,j+1} - 2T_{i,j} + T_{i,j-1})/\Delta y^2 = -q'''/k. \quad (2)$$

The prescribed-temperature boundaries are entered directly into the algebraic system. At the top and bottom surfaces, a ghost-node relation is derived from the convection boundary condition and substituted into the central-difference expression. This retains a second-order interior stencil and incorporates the Robin boundary without introducing additional unknowns. Classical finite-difference heat-transfer studies demonstrate that such structured discretization is efficient when the physical domain is regular (De Vahl Davis, 1983; Mazzarella & Pasini, 2015).

Step 5 - Algebraic Solution and Convergence

All nodal equations are assembled into the sparse linear system $A T = b$, where T is the vector of unknown nodal temperatures. Because the problem is linear under the stated assumptions, the system is solved directly rather than iteratively. For nonlinear extensions involving temperature-dependent properties, radiation, or conjugate flow, an outer iterative loop would be required. More advanced

alternatives such as MLPG, finite pointset, and stabilized finite-element formulations have been reported for complex domains and coupled heat-transfer problems (Li et al., 2011; Dai et al., 2013; Reséndiz-Flores & García, 2014; Huang et al., 2015).

Step 6 - Grid-Independence and Verification

Solutions are computed on 21×21 , 41×41 , and 81×81 grids. The center temperature is used as a local grid-sensitivity metric, while integrated heat rates provide a global conservation check. The total volumetric heat generation per unit out-of-plane thickness is $Q_{gen} = q'''LH = 200 \text{ W m}^{-1}$. Conductive heat rates are integrated along the two isothermal walls, and convective heat losses are integrated along the horizontal boundaries. The residual $|\Sigma Q_{boundary} - Q_{gen}|/Q_{gen}$ is reported as a percentage. This combination of refinement and conservation checking follows the verification principles emphasized in benchmark and error-estimation studies (De Vahl Davis, 1983; Marchi et al., 2013).

Table 2. Computational inputs used for the two-dimensional benchmark.

Parameter	Symbol	Value
Plate width / height	L, H	0.10 m
Thermal conductivity	k	$16 \text{ W m}^{-1} \text{ K}^{-1}$
Volumetric heat generation	q'''	$20,000 \text{ W m}^{-3}$
Left-wall temperature	T_h	373.15 K
Right-wall temperature	T_c	293.15 K
Convection coefficient	h	$15 \text{ W m}^{-2} \text{ K}^{-1}$
Ambient temperature	T_∞	293.15 K
Final computational grid	$N \times N$	81×81 nodes

Results

The computed temperature field is shown in Figure 2. The highest temperature occurs at the prescribed hot wall, while the lowest temperature is fixed at the cold wall. Because the top and bottom boundaries lose heat by convection to the ambient, the isotherms bend toward those surfaces rather than remaining vertically straight. The field therefore confirms that the mixed-boundary formulation produces a two-dimensional thermal response. Internal heat generation raises the interior temperature above the value that would occur for source-free conduction under the same side-wall temperatures.

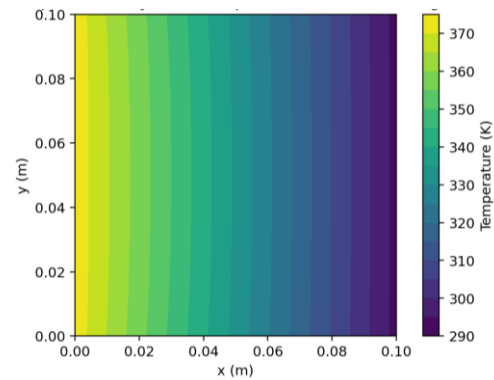


Figure 2. Steady-state temperature contours obtained on the 81×81 finite-difference grid.

Table 3. Grid-independence and energy-balance results.

Grid	Center T (K)	Right-wall heat out (W m^{-1})	Top+bottom convection (W m^{-1})	Energy residual (%)
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21×21	334.056064	1334.98	119.84	2.000
41×41	334.055725	1337.40	119.82	1.000
81×81	334.055639	1338.63	119.82	0.500

The center temperature changes from 334.056064 K on the 21×21 mesh to 334.055725 K on the 41×41 mesh and 334.055639 K on the 81×81 mesh. The change between the two finest grids is only about 0.000026%, indicating that the local temperature solution is effectively grid independent for the reporting precision used here. The integrated energy residual decreases as the grid is refined, reaching approximately 0.50% on the 81×81 grid. The remaining discrepancy is primarily associated with numerical integration and boundary discretization at the corners; it diminishes systematically with refinement.

On the 81×81 grid, the hot left boundary supplies approximately 1259.45 W m⁻¹ to the domain, while the right boundary removes approximately 1338.63 W m⁻¹. The upper and lower convective boundaries each remove about 59.91 W m⁻¹. Since the internally generated 200 W m⁻¹ supplements the incoming heat from the hot side, the total outward heat flow exceeds the left-wall input by nearly the generated amount. This balance is physically consistent with the steady energy equation and provides a stronger validation check than temperature convergence alone.

The results also illustrate why computational heat-transfer methodology should separate model verification from physical interpretation. Grid convergence establishes that the discrete solution is stable with respect to spatial refinement, whereas energy balance checks whether the numerical field satisfies the global conservation law. Similar emphasis on numerical accuracy, efficiency, and verification appears throughout the pre-2016 literature, including meshless transient conduction and stabilized finite-element approaches (Li et al., 2011; Dai et al., 2013; Huang et al., 2015).

References

De Vahl Davis, G. (1983). Natural convection of air in a square cavity: A bench mark numerical solution. *International Journal for Numerical Methods in Fluids*, 3(3), 249–264. <https://doi.org/10.1002/flid.1650030305>

Conclusion and Discussion

A step-by-step finite-difference methodology has been developed for computational analysis of heat transfer in a two-dimensional physical system. The selected benchmark combines internal heat generation, prescribed side-wall temperatures, and convective horizontal boundaries. This combination produces a nontrivial two-dimensional field while remaining sufficiently simple for independent reproduction. The numerical procedure consists of defining the domain and assumptions, writing the energy equation, specifying boundary conditions, discretizing the Poisson equation, assembling and solving a sparse linear system, refining the grid, and verifying global energy conservation. The final 81×81 solution gives a center temperature of approximately 334.056 K, and the grid-refinement study indicates negligible change in that quantity between the two finest meshes. The global energy residual is approximately 0.5%, which is acceptable for the present compact benchmark and decreases with refinement. The method can be extended to anisotropic solids, nonuniform conductivity, transient conduction, irregular boundaries, or coupled conduction-convection systems. For highly irregular geometries or strongly coupled flows, the meshless and stabilized finite-element techniques reported by Reséndiz-Flores and García (2014), Hidayat et al. (2014), and Huang et al. (2015) provide useful alternatives. The principal limitation of the present study is that material properties are constant and the example does not include fluid motion or radiation; therefore, its results should be interpreted as a verified methodological baseline rather than a universal thermal model.

Kandaswamy, P., Lee, J., & Abdul Hakeem, A. K. K. (2007). Natural convection in a square cavity in the presence of heated plate. *Nonlinear Analysis: Modelling and Control*, 12(2), 203–212.

<https://doi.org/10.15388/NA.2007.12.2.1471>
1

Li, Q.-H., Chen, S.-S., & Kou, G.-X. (2011). Transient heat conduction analysis using the MLPG method and modified precise time step integration method. *Journal of Computational Physics*, 230(7), 2736–2750. <https://doi.org/10.1016/j.jcp.2011.01.019>

Selamat, M. S., Hashim, I., & Hasan, M. K. (2012). Transient natural convection in porous square cavity heated and cooled on adjacent walls. *Mathematical Problems in Engineering*, 2012, 253965. <https://doi.org/10.1155/2012/253965>

Marchi, C. H., Novak, L. A., Santiago, C. D., & Vargas, A. P. S. (2013). Highly accurate numerical solutions with repeated Richardson extrapolation for 2D Laplace equation. *Applied Mathematical Modelling*, 37(12–13), 7386–7397. <https://doi.org/10.1016/j.apm.2013.02.043>

Dai, B., Zheng, B., Liang, Q., & Wang, L. (2013). Numerical solution of transient heat conduction problems using improved meshless local Petrov–Galerkin method. *Applied Mathematics and Computation*, 219(19), 10044–10052. <https://doi.org/10.1016/j.amc.2013.04.024>

Reséndiz-Flores, E. O., & García, I. (2014). Application of the finite pointset method to non-stationary heat conduction problems. *International Journal of Heat and Mass Transfer*, 71, 720–723. <https://doi.org/10.1016/j.ijheatmasstransfer.2013.12.077>

Hidayat, M. I. P., Wahjoedi, B. A., Parman, S., & Megat Yusoff, P. S. M. (2014). Meshless local B-spline-FD method and its application for 2D heat conduction problems with spatially varying thermal conductivity. *Applied Mathematics and Computation*, 242, 236–254. <https://doi.org/10.1016/j.amc.2014.05.031>

Huang, P., Zhao, J., & Feng, X. (2015). Highly efficient and local projection-based stabilized finite element method for natural convection problem. *International Journal of Heat and Mass Transfer*, 83, 357–365. <https://doi.org/10.1016/j.ijheatmasstransfer.2014.08.015>

Mazzarella, L., & Pasini, M. (2015). CTF vs FD based numerical methods: Accuracy, stability and computational time's comparison. *Energy Procedia*, 78, 2620–2625. <https://doi.org/10.1016/j.egypro.2015.11.324>